SEMIGLOBAL RESULTS FOR $\overline{\partial}$ ON COMPLEX SPACES WITH ARBITRARY SINGULARITIES, PART II

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Abstract. We obtain some L^2-results for $\overline{\partial}$ on forms that vanish to high order on the singular set of a complex space. As a consequence of our main theorem we obtain weighted L^2-solvability results for compactly supported $\overline{\partial}$-closed (p, q) forms $(0 \le p \le n,\ 1 \le q < n)$ on relatively compact subdomains Ω of the complex space that satisfy $H^{n-q}(\Omega, \mathcal{S}) = 0 = H^{n-q+1}(\Omega, \mathcal{S})$ for every coherent $\mathcal{O}_X$-module $\mathcal{S}$. The latter result can be used to give an alternate proof of a theorem of Merker and Porten.

1. Introduction

This paper is a natural generalization of results obtained in an earlier paper of ours [4]. There, we addressed the question of whether one can solve the equation $\overline{\partial}u = f$ on $\mathrm{Reg}\,\Omega$, the set of smooth points of an open relatively compact *Stein* subset Ω of a pure n-dimensional reduced *Stein* space X, for a $\overline{\partial}$-closed (p, q) form f on $\mathrm{Reg}\,\Omega$ that vanishes to "high order" on the singular set of X. We showed that given any N_0 non-negative integer there exists a positive integer $N = N(N_0,\, p,\, q,\, n,\, \Omega)$ such that for any $\overline{\partial}$-closed form f on $\mathrm{Reg}\,\Omega$, vanishing to order N on $\mathrm{Sing}\,X$, there exists a solution u to $\overline{\partial}u = f$ that vanishes to order N_0 on $\mathrm{Sing}\,X$.

In this paper we consider a pure n-dimensional countable at infinity reduced complex space X (we no longer assume that X is Stein). Let A be a nowhere dense, closed analytic subset of lower dimension containing the singular set $\mathrm{Sing}X$ of X and let Ω be an open, relatively compact subdomain of X (again we do not assume that Ω is Stein). We give $\mathrm{Reg}\,X$ a metric that is compatible with local embeddings and let $|\ |_x$ and dV_x denote the corresponding pointwise norm and volume element. Using a partition of unity argument, we can find a non-negative continuous function d_A on X whose zero set is A with the property that when $X \supset V^{\text{open}} \xrightarrow{\theta} \mathbb{C}^T$ for some positive integer T, is a local embedding, and K is a compact set in V, we have for all $x \in K$

$$d_A(x) \cong \mathrm{dist}\,(\theta(x), \theta(A)).$$

Here the right-hand side distance is defined using the Euclidean metric in $\mathbb{C}^T$ (see section 2 for details on these constructions).

Let us fix $p \in \mathbb{N}$ with $0 \le p \le n$ and let $\mathcal{H}^{s,\mathrm{loc}}_N(\Omega) := \{f \in L^{2,\mathrm{loc}}_{p,\,s}(\Omega \setminus A) : \int_{V\setminus A} |f|^2 d_A^{-N} dV$ is finite for all $V^{\text{open}} \Subset \Omega\}$. Here s is such that $0 \le s \le n$ and N is a non-negative integer. The main result in this paper is the following theorem:

Theorem 1.1. *Let X be a pure n-dimensional (countable at infinity) reduced complex analytic space. Let q be a positive integer with $1 \le q \le n$, and let Ω be a relatively compact subdomain of X such that $H^q(\Omega, \mathcal{S}) = 0$ for all coherent $\mathcal{O}_X$ modules $\mathcal{S}$. For every non-negative integer N_0, there exists a positive integer $N > N_0$ such that: if $f \in \mathcal{H}^{q,loc}_N(\Omega)$ with $\overline{\partial}f = 0$ on $\Omega \setminus A$ then there exists $u \in \mathcal{H}^{q-1,loc}_{N_0}(\Omega)$ with $\overline{\partial}u = f$ on $\Omega \setminus A$.*

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Using the above theorem we obtain weighted L^2-solvability results for compactly supported $\overline{\partial}$-closed forms defined on $\Omega \setminus A$. More precisely we show:

Theorem 1.2. *Let X be as in Theorem 1.1 and let q be a positive integer such that $1 \leq q < n$. Let Ω be an open relatively compact subset of X such that $H^{n-q}(\Omega, \mathcal{S}) = 0 = H^{n-q+1}(\Omega, \mathcal{S})$ for every coherent $\mathcal{O}_X$-module $\mathcal{S}$. Let f be a (p, q) form defined on $\Omega \setminus A$ and $\overline{\partial}$-closed there, compactly supported in Ω and such that $\int_{\Omega\setminus A} |f|^2 \, d_A^{N_0} dV < \infty$ for some $N_0 \geq 0$. Then there exists a $u \in L^{2,loc}_{p,q-1}(\Omega \setminus A)$ with $supp_X\, u$ compact in Ω satisfying $\overline{\partial} u = f$ on $\Omega \setminus A$ and such that $\int_{\Omega\setminus A} |u|^2 \, d_A^N dV < \infty$, where N is a positive constant that depends on N_0 and Ω.*

As a corollary of these solvability results we can obtain an analytic proof of the following theorem of Merker and Porten:

Theorem 1.3. *(Theorem 2.2 in* [13]*) Let X be a connected $(n-1)$-complete normal space of pure dimension $n \geq 2$. Then, for every domain $D \subset X$ and every compact set $K \subset D$ with $D \setminus K$ connected, holomorphic functions on $D \setminus K$ extend holomorphically and uniquely to D.*

Let us point out here that Merker and Porten prove in [13] an extension theorem for meromorphic sections as well. It is not clear to us at the moment how to employ $\overline{\partial}$-techniques to attack the extension problem for such sections. In November of 2008, Coltoiu [3] and Ruppenthal [15] independently obtained a $\overline{\partial}$-theoretic proof of a Hartogs extension theorem on cohomological $(n-1)$-complete (resp. $(n-1)$-complete) spaces. The key ingredient in the proof is a vanishing of the higher direct images of the sheaf of canonical forms of an appropriate desingularization $\tilde{X}$ of the $(n-1)$-complete complex space X. This subtle vanishing result which was obtained by Takegoshi [17], easily yields the vanishing of $H^1_c(\tilde{X}, \mathcal{O})$ which is needed for the Ehrenpreis method to carry over. Our analytic approach is based on more general weighted L^2-solvability results for $\overline{\partial}$-closed, compactly supported (p, q)-forms with $p \neq 0$ which are of independent interest.

The organization of the paper is as follows: In section 2, we recall some of the key lemmata and theorems from [4] and prove the analogous statements under the more general conditions of theorem 1.1. In Section 3, we prove Theorem 1.1. Section 4, contains the weighted L^2-solvability results for $\overline{\partial}$-closed compactly supported forms. In section 5, we outline the analytic proof of the generalized Hartogs extension theorem of Merker-Porten (Theorem 1.3). Section 6 provides a generalization of Lemma 2.1 in [4]. In the short appendix we prove certain lemmata that are used in the construction of d_A.

2. Preliminaries

When X, Ω are Stein (1-complete), Theorem 1.1 was proved in [4]. The proof proceeded along the following lines:

Step 1: Desingularization. We considered $\pi : \tilde{X} \to X$ be a holomorphic surjection with the following properties:

i) $\widetilde{X}$ is an $n-$dimensional complex manifold,
ii) $\tilde{A} = \pi^{-1}(A)$ is a hypersurface in $\tilde{X}$ with only "normal crossing singularities", i.e. near each $x_0 \in \tilde{A}$ there are local holomorphic coordinates $(z_1, \ldots, z_n)$ in terms of which $\tilde{A}$ is given by $h(z) = z_1 \cdots z_m = 0$, where $1 \leq m \leq n$,
iii) $\pi : \widetilde{X} \setminus \tilde{A} \to X \setminus A$ is a biholomorphism,
iv) π is proper.

The existence of such a map follows from the facts that a) every reduced, complex space can be desingularized and, b) every reduced, closed complex subspace of a complex manifold admits an embedded desingularization (the exact statements and proofs can be found in [1], [2]).

Let $\tilde{\Omega} := \pi^{-1}(\Omega)$. We gave $\tilde{X}$ a real analytic metric σ (since by Grauert's result–Theorem 3 in [7]– any real analytic manifold can be properly and analytically embedded in some $\mathbb{R}^T$) and we considered the corresponding distance function $d_{\tilde{A}}(x) = \operatorname{dist}(x, \tilde{A})$, volume element $d\tilde{V}_\sigma$ and norms on $\Lambda^{\cdot} T\tilde{X}$ and $\Lambda^{\cdot} T^*\tilde{X}$.

Let J denote the ideal sheaf of $\tilde{A}$ in $\tilde{X}$ and Ω^p the sheaf of holomorphic p forms on $\tilde{X}$. We considered some auxiliary sheaves (denoted by $\mathcal{L}_{p,q}$) on $\tilde{X}$. For every open subset U of $\tilde{X}$, let $\mathcal{L}_{p,q}(U)$ be

$$\mathcal{L}_{p,q}(U) := \{u \in L^{2,\,\mathrm{loc}}_{p,q}(U);\ \ \overline{\partial}u \in L^{2,\,\mathrm{loc}}_{p,q+1}(U)\} \tag{1}$$

and for each open subset $V \subset U$, let $r^U_V : \mathcal{L}_{p,q}(U) \to \mathcal{L}_{p,q}(V)$ be the obvious restriction maps. Then the map $u \to \overline{\partial}u$ defines an $\mathcal{O}_{\tilde{X}}$-homomorphism $\overline{\partial} : \mathcal{L}_{p,q} \to \mathcal{L}_{p,q+1}$ and the sequence

$$0 \to \Omega^p \to \mathcal{L}_{p,0} \to \mathcal{L}_{p,1} \to \cdots \to \mathcal{L}_{p,n} \to 0$$

is exact by the local Poincaré lemma for $\overline{\partial}$. Since each $\mathcal{L}_{p,q}$ is closed under multiplication by smooth cut-off functions we have a fine resolution of Ω^p. In the same way, since J is locally generated by one function, then the sequence

$$0 \to J^k\Omega^p \to J^k\mathcal{L}_{p,0} \to \cdots \to J^k\mathcal{L}_{p,n} \to 0 \tag{2}$$

is a fine resolution of $J^k\Omega^p$. Here, $u \in (J^k\mathcal{L}_{p,q})_x$ if it can locally be written as $h^k\, u_0$ where h generates J_x and $u_0 \in (\mathcal{L}_{p,q})_x$. So we can interpret the sheaf cohomology groups $H^q(\tilde{\Omega}, (J^k\Omega^p)_{|\Omega})$ as

$$H^q(\tilde{\Omega}, (J^k\Omega^p)_{|\Omega}) \cong \frac{\ker(\overline{\partial} : J^k\mathcal{L}_{p,q}(\tilde{\Omega}) \to J^k\mathcal{L}_{p,q+1}(\tilde{\Omega}))}{\mathrm{Im}(\overline{\partial} : J^k\mathcal{L}_{p,q-1}(\tilde{\Omega}) \to J^k\mathcal{L}_{p,q}(\tilde{\Omega}))}. \tag{3}$$

Step 2: Comparison estimates for pointwise norms of forms and their pullbacks under the desingularization map. Using Lojasiewicz inequalities we proved the following pointwise estimates:

Lemma 2.1. *(Lemma 3.1 in [4]) We have for $x \in \widetilde{\Omega} \setminus \tilde{A}$, $v \in \wedge^r T_x(\widetilde{\Omega})$*

$$c'\ d^t_{\tilde{A}}(x) \quad\leq\quad d_A(\pi(x)) \leq C'\ d_{\tilde{A}}(x), \tag{4}$$
$$c\ d^M_{\tilde{A}}(x)\ |v|_{x,\sigma} \quad\leq\quad |\pi_*(v)|_{\pi(x)} \leq C\ |v|_{x,\sigma} \tag{5}$$

for some positive constants c', c, C', C, t, M, where c, C, M may depend on r.

For an r-form a in $\Omega \setminus A$ set $|\pi^* a|_{x,\sigma} := max\{\ | < a_{\pi(x)}, \pi_* v > |\ ;\ |v|_{x,\sigma} \leq 1,\ v \in \wedge^r T_x(\tilde{\Omega} \setminus \tilde{A})\}$, where by $<,>$ we denote the pairing of an r-form with a corresponding tangent vector. Using (5) we obtain:

$$c\ d^M_{\tilde{A}}(x)\ |a|_{\pi(x)} \leq |\pi^* a|_{x,\sigma} \leq C\ |a|_{\pi(x)} \tag{6}$$

on $\tilde{\Omega}$, for some positive constant M.

Step 3: Cohomological vanishing results. Inspired by Grauert's Satz 1, Section 4 in [6] (Grauert's result corresponds to the case where A is a finite set) we were led to the vanishing of a canonical morphism between certain sheaf cohomology groups on the desingularized space. More precisely we were able to show the following:

Proposition 2.2. *(Proposition 1.3 in [4]) For $q > 0$ and $k \geq 0$ given, there exists a natural number ℓ, $\ell \geq k$ such that the map*

$$i_* : H^q(\tilde{\Omega}, J^\ell\Omega^p) \to H^q(\tilde{\Omega}, J^k\Omega^p),$$

induced by the inclusion $i : J^\ell\,\Omega^p \to J^k\,\Omega^p$, is the zero map.

The above Proposition played a key role in the proof of Theorem 1.1, when X, Ω were Stein. Here is an outline for the proof of that theorem. Given N_0 we chose appropriately k, N ($k \geq M + t\frac{N_0}{2}$, $N \geq 2n\ell + M_1$ where M and t are the exponents that appear in formulas (4) and (6) of lemma 2.1, ℓ is the integer that appears in Proposition 2.2 and M_1 is the exponent that appears on the left-hand side of formula (6) for the pull-back of the volume form). Then using a change of variables formula (lemma 4.1 in [4]) we had:

$$\int_{\tilde{\Omega}} |\pi^* f|^2_\sigma d_{\tilde{A}}^{-N_1} d\tilde{V}_\sigma \leq C \int_{\Omega\setminus A} |f|^2 d_A^{-N} dV, \tag{7}$$

for a suitable $0 < N_1 := N - M_1 < N$ and as a consequence we obtained that $\overline{\partial}\pi^* f = 0$ on $\tilde{\Omega}$.
Using (7) and the way we had chosen N, k we showed that $\pi^* f \in J^\ell \mathcal{L}_{p,q}(\tilde{\Omega})$. By Proposition 2.2, this implied that the equation $\overline{\partial} v = \pi^* f$ had a solution in $J^k \mathcal{L}_{p,q-1}(\tilde{\Omega})$. Since $|h(x)| \leq C d_{\tilde{A}}(x)$ on compacts in the set where h generates J it followed that

$$\int_{\widetilde{\Omega}'} |v|^2_\sigma d_{\tilde{A}}^{-2k}(x) d\tilde{V}_\sigma < \infty$$

where $\widetilde{\Omega}' = \pi^{-1}(\Omega')$ and $\Omega' \subset\subset \Omega$. Then $\overline{\partial}((\pi^{-1})^* v) = f$ on $\Omega \setminus A$ and the final step was to show that

$$\int_{\Omega'\setminus A} |(\pi^{-1})^* v|^2 d_A^{-N_0} dV \leq c \int_{\widetilde{\Omega}'} |v|^2_\sigma d_{\tilde{A}}^{-2k} d\tilde{V}_\sigma. \tag{8}$$

2.1. **Necessary modifications for the general case.** When X, Ω are no longer Stein (and hence can not be thought of as subvarieties embedded in some $\mathbb{C}^T$) we have to work with locally finite Stein coverings of X, Ω and carefully modify the previous approach.

Notation: Let $a,\ b$ be two non-negative functions defined on a set E. We shall write $a \sim_E b$ iff there exists a positive constant $C \geq 1$ such that $a \leq C\, b$ and $b \leq C\, a$ on E.

Clearly $\sim_E$ is an equivalence relation on the set of non-negative functions defined on E. Also, it is not hard to show that if $\{E_j\}_{j=1}^M$ is a finite covering of E and $f,\ g$ are non-negative functions on E then $f \sim_E g$ iff $f \sim_{E_j} g$ for all $j = 1, \cdots, M$.

Construction of the distance function d_A.

Proposition 2.3. *Let X be a complex space and A be a subvariety containing Sing$X$. Then, there exists a non-negative function d_A defined on X, with zero set A that satisfies the following:*

If $\phi : U \to C^T$ is a local embedding of $U^{open} \subset X$ and $K,\ K'$ compact subsets of U with $K \Subset K' \Subset U$, we have

$$d_A(x) \sim_K \mathit{dist}(\phi(x),\ \phi(K' \cap A)) \tag{9}$$

where the right-hand side distance is the Euclidean distance from the point $\phi(x)$ to the closed set $\phi(K' \cap A)$ in $\mathbb{C}^T$.

Before we prove Proposition 2.3 let us make some remarks: The right-hand side of (9) can be thought of as the local definition of d_A. There are two things one needs to worry about: a) How does this local definition depend on the compact $K' \Supset K$ that contains K? b) How does this definition depend on the embedding? With a little bit of work (see section 7) one can check that the distances we would get by choosing a different compact L such that $L \Supset K$ or a different embedding would be equivalent to each other over K.

Proof: We choose local embeddings $\phi_i : U_i' \to \mathbb{C}^{T_i}$ where U_i' are open charts in X. Let U_i be open relatively compact subsets of U_i' for $i = 1, 2 \cdots$ with $U_i \cap A \neq \emptyset$ and such that $\{U_i\}_{i\geq 1}$ form a locally finite covering of A. We choose an open subset U_0 of X with $\overline{U}_0 \cap A = \emptyset$ and such that $\{U_i\}_{i\geq 0}$ form a locally finite covering of X. Let $\{\psi_i\}_{i\geq 0}$ be a smooth partition of unity subordinate to the covering $\{U_i\}_{i\geq 0}$ (i.e. $0 \leq \psi_i \leq 1$, $\operatorname{supp} \psi_i \subset U_i$ for all i and such that $\sum_{i=0}^\infty \psi_i = 1$).

Then we define

$$d_A(x) := \psi_0(x) + \sum_{i=1}^{\infty} \psi_i(x)\, \mathrm{dist}(\phi_i(x), \phi_i(\overline{U}_i \cap A)). \tag{10}$$

The i-th right-hand side term in (10) is extended by 0 outside $\operatorname{supp}\psi_i$. Clearly d_A is a non-negative, continuous function on X whose zero set $d_A^{-1}(0) = A$.

It remains to show that d_A satisfies (9). Let U, K, K' be as in Proposition 2.3. Set $K_j := K \cap \operatorname{supp}\psi_j$ and $K'_j := \overline{U}_j \cap K'$. Since $\{U_j\}_{j\geq 0}$ is a locally finite covering of X there would exist finitely many $j's$ such that $K \cap \operatorname{supp}\psi_j \neq \emptyset$. If $x \in K \cap \operatorname{supp}\psi_0$, we have

$$0 < a_0 \leq \mathrm{dist}(\phi(x),\, \phi(K' \cap A)) \leq A_0. \tag{11}$$

From the above inequalities one can conclude that when $x \in K \cap \operatorname{supp}\psi_0$ we have that

$$\psi_0(x) \leq 1 \leq a_0^{-1}\, \mathrm{dist}(\phi(x),\, \phi(K' \cap A)). \tag{12}$$

On the other hand, if $x \in K \cap \operatorname{supp}\psi_j$ with $j > 0$ we have the following:

$$\begin{aligned}
\mathrm{dist}(\phi(x),\, \phi(K' \cap A)) \quad &\sim_{K_j} \quad \mathrm{dist}(\phi(x),\, \phi(K'_j \cap A)) \\
&\sim_{K_j} \quad \mathrm{dist}(\phi_j(x),\, \phi_j(K'_j \cap A)) \quad \text{(as different embeddings yield equivalent norms)} \\
&\sim_{K_j} \quad \mathrm{dist}(\phi_j(x),\, \phi_j(\overline{U}_j \cap A)).
\end{aligned}$$

Let $\{C_j\}_{j>0}$ denote the positive constants that arise from the fact that

$$\mathrm{dist}(\phi(x),\, \phi(K' \cap A)) \sim_{K_j} \mathrm{dist}(\phi_j(x),\, \phi_j(\overline{U}_j \cap A))$$

i.e. $C_j \geq 1$ and for all $x \in K_j$ we have $\mathrm{dist}(\phi(x),\, \phi(K' \cap A)) \leq C_j\, \mathrm{dist}(\phi_j(x),\, \phi_j(\overline{U_j} \cap A))$ and $\mathrm{dist}(\phi_j(x),\, \phi_j(\overline{U}_j \cap A)) \leq C_j\, \mathrm{dist}(\phi(x),\, \phi(K' \cap A))$.

Then, for $x \in K$ we have:

$$\begin{aligned}
d_A(x) \quad &= \quad \psi_0(x) + \sum_{i=1}^{\infty} \psi_i(x)\, \mathrm{dist}(\phi_i(x), \phi_i(\overline{U}_i \cap A)) \\
&\leq \quad a_0^{-1}\, \mathrm{dist}(\phi(x),\, \phi(K' \cap A)) + \sum_{\psi_j(x)\neq 0} C_j\, \psi_j(x)\, \mathrm{dist}(\phi(x),\, \phi(K' \cap A)) \\
&\leq \quad C\, \mathrm{dist}(\phi(x), \phi(K' \cap A))
\end{aligned}$$

where $C := \max\{a_0^{-1}, A_0, \sum_{j>0; K\cap \operatorname{supp}\psi_j \neq \emptyset} C_j\}$. On the other hand for $x \in K$ we have:

$$\begin{aligned}
\mathrm{dist}(\phi(x),\, \phi(K' \cap A)) \quad &= \quad \sum \psi_j(x)\, \mathrm{dist}(\phi(x),\, \phi(K' \cap A)) \\
&\leq \quad A_0\, \psi_0(x) + \sum_{j>0; \psi_j(x)\neq 0} C_j \psi_j(x)\, \mathrm{dist}(\phi_j(x), \phi_j(\overline{U}_j \cap A)) \\
&\leq \quad C\, d_A(x).
\end{aligned}$$

2.2. **A Hermitian metric on Reg X.** We give Reg X a hermitian metric $\tilde{\sigma}$ compatible with local embeddings and let $|\ |_{\tilde{\sigma}}$ denote the pointwise norms on tangent vectors, cotangent vectors induced by σ. Compatible with local embeddings means the following: If $\phi : U \to \mathbb{C}^T$ is a local embedding of X and K a compact subset of U we have for all $v \in T_z(\mathrm{Reg}X)$ with $z \in K \cap \mathrm{Reg}\, X$

$$|\phi_*(v)|_E \sim |v|_{\tilde{\sigma}}.$$

By $|\ |_E$ we mean the pointwise norm induced by the Euclidean metric in $\mathbb{C}^T$.

2.3. **Construction of special cut-off functions.** In section 4, we shall need the existence of special cut-off functions. Using the same notation as above we have:

Proposition 2.4. *For each $\nu \in \mathbb{N}$ there exists a function $\chi_\nu \in C^\infty(X; [0,1])$ such that $\chi_\nu(x) = 1$ when $d_A(x) \geq \frac{1}{\nu}$ and $\chi_\nu(x) = 0$ near A. Furthermore for every K compact subset of X there exists a constant C_K such that*

$$|\overline{\partial}\chi_\nu(x)|_{\tilde{\sigma}} \leq C_K\, \nu,$$

for all $x \in K \cap \mathrm{Reg}\, X$.

Proof: The proof is based on the following well known principle:

Lemma 2.5. *For every $d \in \mathbb{N}$, there exists a positive constant C_d with the following property: For every closed subset F of $\mathbb{R}^d$ and $\nu \in \mathbb{N}$ there exists $\chi_{\nu,F} \in C^\infty(\mathbb{R}^d; [0,1])$ such that $\chi_{\nu,F}(x) = 1$ when dist$(x, F) \geq \frac{1}{\nu}$, $\chi_{\nu,F} = 0$ near F and $|d\chi_{\nu,F}|_E \leq C_d\, \nu$.*

We consider the family $(\phi_j, U_j', U_j, \psi_j)$ from the proof of Proposition 2.3 in section 2.1. For $j > 0$ taking as $K := \mathrm{supp}\psi_j$, $K' := \overline{U}_j$, $U := U_j'$, $\phi := \phi_j$ we obtain from (9) the existence of positive integers m_j such that whenever $\mathrm{dist}(\phi_j(x),\, \phi_j(\overline{U_j} \cap A)) < \frac{\delta}{m_j} \Rightarrow d_A(x) < \delta$ on $\mathrm{supp}\, \psi_j$.

For each ν, $j \in \mathbb{N}$ we define (using lemma 2.5)

$$\chi_{j,\,\nu} := \chi_{m_j\,\nu,\, \phi_j(\overline{U_j} \cap A)}$$

on $\mathbb{C}^{T_j}$ and we set

$$\chi_\nu(x) := \psi_0(x) + \sum_{j=1}^{\infty} \psi_j(x)\, \chi_{j,\,\nu}(\phi_j(x)) \tag{13}$$

Clearly $\chi_\nu \in C^\infty(X; [0,1])$ and $\chi_\nu = 0$ near A. Whenever $d_A(x) \geq \frac{1}{\nu}$, we must have $\chi_{j,\nu}(x) = 1$ when $x \in \mathrm{supp}\, \psi_j$, since $\mathrm{dist}(\phi_j(x),\, \phi_j(\overline{U_j} \cap A)) \geq \frac{1}{m_j\,\nu}$. Hence $\chi_\nu(x) = \sum_{j=0}^{\infty} \psi_j(x) = 1$. Moreover,

$$\overline{\partial}\chi_\nu = \overline{\partial}\psi_o + \sum_{j=1}^{\infty} \left(\overline{\partial}\psi_j \wedge \chi_{j,\nu} + \psi_j\, \overline{\partial}\chi_{j,\nu}\right).$$

Given any smooth function f on X we have that $|\overline{\partial} f|_{\tilde{\sigma}}$ is bounded on $K \cap \mathrm{Reg}\, X$ for K compact subset of X. Also, given any K compact subset of X there are finitely many j's such that $K \cap \mathrm{supp}\, \psi_j \neq \emptyset$. Hence for $x \in K \cap \mathrm{Reg}\, X$ we have

$$|\overline{\partial}\chi_\nu|_{\tilde{\sigma}} \leq C_K' + \sum_{j;\mathrm{supp}\psi_j \cap K \neq \emptyset} \psi_j |\phi^*(\overline{\partial}\chi_{j,\nu})|_{\tilde{\sigma}} \tag{14}$$

$$\leq C_K' + \nu C_K'' \leq (C_K' + C_K'')\, \nu. \tag{15}$$

In the last estimate we used the fact that $|\overline{\partial}\chi_{j,\,\nu}|_E \leq C_{2T_j}\, m_j\, \nu$ and that $|\phi_j^*(\theta)|_{\tilde{\sigma}} \leq C_j'\, |\theta|_E$ on $\mathrm{supp}\, \psi_j$.

2.4. **Generalization of results obtained in sections 3 and 4 in** [4]**.** Let X be a pure n-dimensional complex space countable at infinity, let Ω be an open relatively compact subdomain of X and let $\pi : \tilde{X} \to X$ be a desingularization map as in Step 1, of section 2. Let d_A and $\tilde{\sigma}$ be the distance function to A and the Hermitian metric on $\operatorname{Reg} X$ that were constructed in sections 2.1-2.2. The pointwise comparison estimates of forms and their pull-backs under the desingularization map π (Lemma 2.1) as well as a change of variables formula that were obtained in [4] carry over to the more general situation we consider in this paper, using a finite Stein covering of $\overline{\Omega}$. More precisely we have:

Lemma 2.6. *For every $t \in \mathbb{N}^*$, there exists $N \in \mathbb{N}$ such that if $f \in \mathcal{H}^{s,\,loc}_N(\Omega)$ then $\pi^* f \in J^t\,\mathcal{L}_{p,s}(\tilde{\Omega})$.*

Lemma 2.7. *For every non-negative integer N_0, there exists $t_0 \in \mathbb{N}^*$ such that if $\tilde{u} \in J^{t_0}\,\mathcal{L}_{p,s}(\tilde{\Omega})$ then $(\pi^{-1})^*(\tilde{u}) \in \mathcal{H}^{s,\,loc}_{N_0}(\Omega)$.*

Uniformity of exponents in Lemmata 2.6, 2.7. A careful inspection of the proofs of Lemmata 3.1 and 4.1 in [4] show that for fixed p, s, N_0, t the exponents N and t_0 that appear in Lemmata 2.6, 2.7 can also be used for forms $f \in \mathcal{H}^{s,\,loc}_N(W)$ or $\tilde{u} \in J^{t_0}\,\mathcal{L}_{p,s}(\tilde{W})$ where W is *any* open subset of Ω.

2.5. **Generalization of Proposition 2.2.** Let X be a pure n-dimensional complex space countable at infinity and let Ω be an open relatively compact subdomain of X and let $\pi : \tilde{X} \to X$ be a desingularization map as in Step 1, of section 2. Then we have the following

Proposition 2.8. *For every $t_0 \in \mathbb{N}^*$ there exists a positive integer $t \geq t_0$ such that the map*

$$i_* : R^s\pi_*(J^t\,\Omega^p_{\tilde{X}}) \to R^s\pi_*(J^{t_0}\Omega^p_{\tilde{X}})$$

induced by the inclusion homomorphism $i : J^t\Omega^p_{\tilde{X}} \to J^{t_0}\Omega^p_{\tilde{X}}$, is the zero homomorphism over Ω, for every $s \in \mathbb{N}^*$.

Proof: We can find finitely many Stein subdomains $\Omega_j \subset\subset \Omega_j^* \subset\subset X$ with $j = 1, \cdots, N_1$ that cover Ω. From the proof of Proposition 2.2 (Proposition 1.3 in [4]) it is clear that for any $t_0 \geq 0$ integer and for any coherent analytic $\mathcal{O}_{\tilde{X}}$-module $\mathcal{S}$, there exists a positive integer $t_j \geq t_0$ such that $i_* : H^s(\tilde{\Omega}_j,\, J^{t_j}\mathcal{S}) \to H^s(\tilde{\Omega}_j,\, J^{t_0}\,\mathcal{S})$ is the zero map for $s > 0$. Take as $\mathcal{S} := \Omega^p_{\tilde{X}}$. By Satz 5, Section 2 in [5] since each Ω_j is Stein, we know that

$$H^s(\tilde{\Omega}_j,\, J^{t_j}\,\Omega^p_{\tilde{X}}) \cong H^0(\Omega_j,\, R^s\,\pi_*(J^{t_j}\,\Omega^p_{\tilde{X}})). \tag{16}$$

Combining these observations together we conclude that $H^0(\Omega_j,\, R^s\pi_*(J^{t_j}\,\Omega^p_{\tilde{X}})) \to H^0(\Omega_j,\, R^s\pi_*(J^{t_0}\Omega^p_{\tilde{X}}))$ is the zero map. By Cartan's theorem A (Chapter I, vol. III in [10]) we know that for every $x \in \Omega_j$ the germs at x of global sections $H^0(\Omega_j,\, R^s\,\pi_*(J^{t_j}\,\Omega^p_{\tilde{X}}))$ generate the stalk $(R^q\pi_*(J^{t_j}\Omega^p_{\tilde{X}}))_x$. Hence the induced map $i_* : R^s\pi_*(J^{t_j}\Omega^p_{\tilde{X}}) \to R^s\pi_*(J^{t_0}\Omega^p_{\tilde{X}})$ is the zero homomorphism over each Ω_j since a set of generators of each stalk is mapped to zero. Let us set $t := \max(t_1,\, t_2, \cdots, t_{N_1})$. Then $i_* : R^s\pi_*(J^t\Omega^p_{\tilde{X}}) \to R^s\pi_*(J^{t_0}\Omega^p_{\tilde{X}})$ is the zero map over each Ω_j and hence over Ω.

If W is a Stein subset of Ω we see that

$$H^s(\tilde{W},\, J^t\Omega^p_{\tilde{X}}) \xrightarrow{i_*} H^s(\tilde{W},\, J^{t_0}\Omega^p_{\tilde{X}})$$

is the zero map for all $s > 0$. This follows from the following commutative diagram:

$$\begin{array}{ccc} H^s(\tilde{W},\, J^t\Omega^p_{\tilde{X}}) & \xrightarrow{\quad i_* \quad} & H^s(\tilde{W},\, J^{t_0}\Omega^p_{\tilde{X}}) \\ \cong\big\downarrow & & \cong\big\downarrow \\ H^0(W,\, R^s\pi_*(J^t\Omega^p_{\tilde{X}})) & \xrightarrow{i_*} & H^0(W,\, R^s\pi_*(J^{t_0}\Omega^p_{\tilde{X}})) \end{array} .$$

The vertical map are isomorphisms by Satz 5, section 2 in [5] and the bottom horizontal map is zero from the result of the previous paragraph.

Uniformity of exponents in Proposition 2.8: As Ω_j are Stein, by Cartan's theorem A we know that for any coherent $\mathcal{O}_X$ module $\mathcal{G}$, the space $H^0(\Omega_j, \mathcal{G})$ generates $\mathcal{G}_x$ for each $x \in \Omega_j$. If $W_j \subset \Omega_j$ Stein subdomains of Ω_j for each fixed s we will still have that $i_* : R^s\, \pi_*(J^{t_j}\mathcal{S}) \to R^s\, \pi_*(J^{t_0}\mathcal{S})$ is the zero map over W_j and $i_* : H^s(\tilde{W}_j,\, J^{t_j}\, \mathcal{S}) \to H^s(\tilde{W}_j,\, J^{t_0}\, \mathcal{S})$ is the zero map for the same values of t_j (or even t) as above.

Now using Proposition 2.8, for $q > 0$ given and t_0 given we will inductively define positive integers $t_1,\, t_2, \cdots, t_q$ with $t_0 \leq t_1 \leq \cdots \leq t_q$ such that

$$R^s\pi_*(J^{t_{j+1}}\Omega^p_{\tilde{X}}) \to R^s\pi_*(J^{t_j}\Omega^p_{\tilde{X}}) \tag{17}$$

induced by the inclusion map $i : J^{t_{j+1}}\Omega^p_{\tilde{X}} \to J^{t_j}\Omega^p_{\tilde{X}}$, is the zero map over Ω for $s > 0$ and for all j with $0 \leq j \leq q-1$.

3. Proof of Theorem 1.1

Let $\mathcal{U} = \{U_j\}_{j\in J}$ be a locally finite Stein covering of Ω and let $\tilde{\mathcal{U}} = \{\tilde{U}_j := \pi^{-1}(U_j)\}_{j\in J}$ be a locally finite covering of $\tilde{\Omega} = \pi^{-1}(\Omega)$. Let r, s, t be non-negative integers. We introduce the spaces of alternating r co-chains

$$C^{r,s}_t := C^r(\tilde{\mathcal{U}}, J^t\mathcal{L}_{p,s}).$$

Here $\mathcal{L}_{p,s}$ are the sheaves defined in (1). Let $\delta : C^{r,s}_t \to C^{r+1,s}_t$ be the co-boundary map and $\overline{\partial} : C^{r,s}_t \to C^{r,s+1}_t$ the $\overline{\partial}$ operator applied componentwise, i.e. given any $c = \{c_{j_0 j_1 \cdots j_r}\}$ we define $\overline{\partial}c$ to be the r co-chain such that $(\overline{\partial}c)_{j_0 j_1 \cdots j_r} = \overline{\partial}c_{j_0 j_1 \cdots j_r}$. Clearly $\delta \circ \overline{\partial} = \overline{\partial} \circ \delta$. Set $U_{j_0 j_1 \cdots j_r} := U_{j_0} \cap U_{j_1} \cap \cdots \cap U_{j_r}$ (as U_j are Stein, $U_{j_0 j_1 \cdots j_r}$ are Stein).

Let N_0, q be as in Theorem 1.1. Using Lemma 2.7 we can find an integer t_0 such that if $\tilde{u} \in J^{t_0}\, \mathcal{L}_{p,q-1}(\tilde{\Omega})$ then $(\pi^{-1})^*(\tilde{u}) \in \mathcal{H}^{q-1,\, loc}_{N_0}(\Omega)$. With t_0 chosen as before let us choose $\{t_j\}_{1\leq j\leq q}$, positive integers with $t_0 \leq t_1 \leq \cdots \leq t_q$ in such a way such that the maps determined by (17) are the zero maps over Ω for all $0 \leq j < q$.

Claim 3.1. *Given any $c \in C^{r,s}_{t_{j+1}}$ with $s > 0$ and $\overline{\partial}c = 0$ we can find a solution to $\overline{\partial}d = c$ with $d \in C^{r,s-1}_{t_j}$ for $0 \leq j < q$.*

Proof of Claim: The Claim follows easily from Step 1 in section 2.1 taking into account the facts that $U_{j_0 j_1 \cdots j_r}$ are Stein, (3) holds and that the maps in (17) are zero over Ω for all j with $0 \leq j \leq q-1$. □

Now, with q and t_q chosen as above, Lemma 2.6 determines an integer N such that if $f \in \mathcal{H}^{q,\, loc}_N(\Omega)$ with $\overline{\partial}f = 0$ on $\Omega \setminus A$ then $\pi^*f \in J^{t_q}\, \mathcal{L}_{p,q}(\tilde{\Omega})$ and $\overline{\partial}\pi^*f = 0$ on $\tilde{\Omega}$.

The proof of Theorem 1.1 will proceed as follows:

Part A: To π^*f we will associate a q co-cycle $c^q \in Z^q(\tilde{\mathcal{U}},\, J^{t_0}\Omega^p_{\tilde{X}})$ (in a similar way as when we prove the quasi-isomorphism between Dolbeault cohomology and Čech-cohomology on a manifold).

Using π^*f we define a 0 co-cycle $c^0 := \{c^0_j\}$ such that $c^0 \in C^{0,q}_{t_q}$ by setting $c^0_j := \pi^*f_{\restriction(\tilde{U}_j\setminus\tilde{A})}$ for all $j \in \mathbb{N}$. Clearly $\delta c^0 = 0$ and $\overline{\partial}c^0 = 0$.

Using Claim 3.1, we can inductively define r co-chains $c^r \in C^{r,q-r}_{t_{q-r}}$ for $1 \leq r \leq q$ and $d^r \in C^{r,q-r-1}_{t_{q-r-1}}$ for $0 \leq r < q$ such that

(i) $\overline{\partial}d^r = c^r$ for all r with $0 \leq r \leq q-1$,

(ii) $c^{r+1} = \delta d^r$ for all r with $0 \le r \le q-1$,

(iii) $\delta c^r = 0$ and $\overline{\partial} c^r = 0$ for all r with $0 \le r \le q$.

Suppose that $c^0, c^1, \cdots, c^r$ and $d^0, d^1, \cdots, d^r$ with $0 \le r \le q-1$ satisfying (i)-(iii) are given. Set $c^{r+1} := \delta d^r$. Clearly $\delta c^{r+1} = 0$ and $\overline{\partial} c^{r+1} = \overline{\partial}\delta d^r = \delta\overline{\partial} d^r = \delta c^r = 0$. If $r+1 < q$ then by Claim 3.1, we can find d^{r+1} such that $\overline{\partial} d^{r+1} = c^{r+1}$. Then c^{r+1}, d^{r+1} satisfy (i)-(iii) and the procedure is completed. If $r+1 = q$ then the previous procedure yielded an element $c^q = \delta d^{q-1} \in C^{q,\,0}_{t_0}$ such that $\overline{\partial} c^q = 0$. But the latter implies that c^q is a co-cycle in $C^q(\tilde{\mathcal{U}},\, J^{t_0}\,\Omega^p_{\tilde{X}})$.

Part B: Taking into account that for any coherent analytic sheaf $\mathcal{S}$ on $\tilde{X}$ and for any U open in X we have $\pi_*\mathcal{S}(U) = \mathcal{S}(\tilde{U})$, we can see that c^q will define a q co-cycle $\overline{c}^q$ in $C^q(\mathcal{U},\, \pi_* J^{t_0}\Omega^p_{\tilde{X}})$. Since $\mathcal{U}$ is a Stein covering of Ω we have that $H^q(\mathcal{U},\, \pi_*(J^{t_0}\Omega^p_{\tilde{X}})) \cong H^q(\Omega, \pi_*(J^{t_0}\Omega^p_{\tilde{X}}))$. But the latter group vanishes since $\pi_*(J^{t_0}\Omega^p_{\tilde{X}})$ is a coherent sheaf on X and by assumption $H^q(\Omega,\, \mathcal{S}) = 0$ for all coherent $\mathcal{O}_X$-modules $\mathcal{S}$. Hence there exists a $\tilde{h}^{q-1} \in C^{q-1}(\mathcal{U}, \pi_*(J^{t_0}\Omega^p_{\tilde{X}}))$ such that $\delta\overline{h}^{q-1} = \overline{c}^q$ and thus there exists an $h^{q-1} \in C^{q-1}(\tilde{\mathcal{U}}, J^{t_0}\Omega^p_{\tilde{X}})$ such that $\delta h^{q-1} = c^q$. We set $d'^{q-1} := d^{q-1} - h^{q-1}$. Clearly $\delta d'^{q-1} = 0$ and $\overline{\partial} d'^{q-1} = c^{q-1}$.

If $q = 1$ then we are done. If $q > 1$ then by downward induction on r, we shall modify d^r to $d'^r \in C^{r,\,q-r-1}_{t_0}$ satisfying for all r with $0 \le r \le q-1$

$$\delta d'^r = 0 \quad \text{and} \quad \overline{\partial} d'^r = c^r. \tag{18}$$

We start with d'^{q-1} as above. Suppose d'^j are given for $j = q-1,\, q-2, \cdots r$ (with $r > 0$) satisfying (18). As $\delta d'^r = 0$ we will construct a $(q-1)$-co-chain $a^{r-1} \in C^{r-1,\,q-r-1}_{t_0}$ that satisfies $\delta a^{r-1} = d'^r$. Let $\{\phi_j\}$ be a smooth partition of unity subordinate to the cover $\tilde{\mathcal{U}}$ of $\tilde{\Omega}$. We define an $(r-1)$ co-chain $a^{r-1} \in C^{r-1,\,q-r-1}_{t_0}$ by setting:

$$(a^{r-1})_{j_0 j_1 \cdots j_{r-1}} := \sum \phi_j d'^{\,r}_{j j_0 \cdots j_{r-1}} \tag{19}$$

It is a standard fact that $\delta a^{r-1} = d'^r$. Taking $\overline{\partial}$ on both sides of the above equation we obtain $\overline{\partial}\delta a^{r-1} = \overline{\partial} d'^r = c^r = \delta d^{r-1}$. Hence we have $\delta(d^{r-1} - \overline{\partial} a^{r-1}) = 0$. Let us set $d'^{r-1} := d^{r-1} - \overline{\partial} a^{r-1}$. Then $\overline{\partial} d'^{r-1} = c^{r-1}$ and $\delta d'^{r-1} = 0$. It remains to show that $d'^{r-1} = d^{r-1} - \overline{\partial} a^{r-1} \in C^{r-1,q-r}_{t_0}$ for the induction step to be completed. From (19) we obtain:

$$\overline{\partial} a^{r-1}_{j_0 \cdots j_{r-1}} = \sum_j \overline{\partial}\phi_j \wedge d'^r_{j j_0 \cdots j_{r-1}} + \sum \phi_j c^r_{j j_0 \cdots j_{r-1}} \in J^{t_0}\mathcal{L}_{p,q-r},$$

as $|\overline{\partial}\phi_j|_{\tilde{\sigma}}$ is bounded for each j and $\{\operatorname{supp}\phi_j\}_j$ is locally finite. Hence $d'^r \in C^{r-1,q-r}_{t_0}$.

Hence we can find $d'^0 \in C^{0,q-1}_{t_0}$ with $\delta d'^0 = 0$ and $\overline{\partial} d'^0 = c^0$. But $\delta d'^0 = 0$ implies that $d'^{\,0}_j = d'^{\,0}_k$ on $\tilde{U}_j \cap \tilde{U}_k$ (when non-empty). Hence $d'^0 \in C^{0,q-1}_{t_0}$ defines a form $\tilde{u} \in J^{t_0}\mathcal{L}_{p,q-1}(\tilde{\Omega})$ such that $\overline{\partial}\tilde{u} = \pi^* f$ on $\tilde{\Omega} \setminus \tilde{A}$. Then $u := (\pi^{-1})^*\tilde{u} \in \mathcal{H}^{q-1,\,loc}_{N_0}$ (by Lemma 2.7) and $\overline{\partial} u = f$ on $\Omega \setminus A$.

Remark: If $W \Subset \Omega$ open subdomain of Ω such that $H^q(W, \mathcal{S}) = 0$ for all coherent O_X-modules $\mathcal{S}$, then for any N_0 non-negative integer and any $f \in \mathcal{H}^{q,\mathrm{loc}}_N(W)$ with $\overline{\partial} f = 0$ on $W \setminus A$ there exists $u \in \mathcal{H}^{q-1,\mathrm{loc}}_{N_0}(W)$ with $\overline{\partial} u = f$ on $W \setminus A$. Here N is the same integer as the one that appears in Theorem 1.1. This follows from the uniformity of exponents in Lemmata 2.6, 2.7 and Proposition 2.8, along with the fact that the proof of Claim 3.1 carries over verbatim for such a W.

4. Weighted L^2-solvability results for $\overline{\partial}$-closed compactly supported forms

In [14] (Section 5) we developed an analytic approach to obtaining Hartogs extension theorems on normal Stein spaces (of pure dimension $n \geq 2$) with arbitrary singularities. One of the key elements in that approach was obtaining weighted L^2-solvability results for $\overline{\partial}$-closed compactly supported forms. More precisely we proved the following (under the assumption that X, Ω are 1-complete spaces and for $A := \operatorname{Sing} X$):

Theorem 4.1. *(Theorem 5.3 in* [14]*) Let f be a (p,q) form defined on $Reg\,\Omega$ and $\overline{\partial}$-closed there with $0 < q < n$, compactly supported in Ω and such that $\int_{Reg\,\Omega} |f|^2\, d_A^{N_0} dV < \infty$ for some $N_0 \geq 0$. Then there exists a solution u to $\overline{\partial}u = f$ on $Reg\,\Omega$ satisfying $supp_X\, u \Subset \Omega$ and such that*

$$\int_{Reg\,\Omega} |u|^2\, d_A^{N} dV \leq C \int_{Reg\,\Omega} |f|^2\, d_A^{N_0} dV$$

N depends on N_0 and Ω and C is a positive constant that depends on N_0, N, Ω and $supp\, f$.

Let X now be a pure n-dimensional complex space reduced and countable at infinity, q be a positive integer with $q < n$ and $\Omega \Subset X$ open such that $H^{n-q}(\Omega, \mathcal{S}) = 0 = H^{n-q+1}(\Omega, \mathcal{S})$ for every coherent $\mathcal{O}_X$ module $\mathcal{S}$ and let A be a nowhere dense, lower dimensional complex analytic subset of X containing $\operatorname{Sing} X$. Let $\mathcal{U} := \{U_i\}$ be a locally finite Stein cover of Ω, consisting of open relatively compact Stein subsets of Ω such that each U_i is relatively compact in U_i' (the local coordinate charts (ϕ_i, U_i') that were described in section 2.2). Let $\{\psi_i\}$ be a smooth partition of unity subordinate to this cover. Using local embeddings we define the distance d_A as in section 2.1. We give $\operatorname{Reg} X$ the metric $\tilde{\sigma}$ which was defined in section 2.2. This metric is compatible with local embeddings and for simplicity let $|\,|_x$ and dV_x denote the corresponding pointwise norm and volume element with respect to this metric. Then we can prove the following generalization of Theorem 4.1:

Theorem 4.2. *Let X, Ω, A, q be as above. Let f be a (p,q) form defined on $\Omega \setminus A$ and $\overline{\partial}$-closed there, compactly supported in Ω and such that $\int_{\Omega\setminus A} |f|^2\, d_A^{N_0} dV < \infty$ for some $N_0 \geq 0$. Then there exists a $u \in L^{2,loc}_{p,\,q-1}(\Omega\setminus A)$ with $supp_X\, u$ compact in Ω satisfying $\overline{\partial}u = f$ on $\Omega\setminus A$ and such that $\int_{\Omega\setminus A} |u|^2\, d_A^{N} dV < \infty$, where N is a positive constant that depends on N_0 and Ω.*

Proof: The proof follows along the same lines as the proof of Theorem 4.1 (Theorem 5.3 in [14]). Let us use the symbol $\mathcal{H}_N^{s,\,loc}(\Omega) := \{f \in L^{2,\,loc}_{n-p,\,s}(\Omega \setminus A);\ \int_{V\setminus A} |f|^2\, d_A^{-N}\, dV < \infty \text{ for all } V \Subset \Omega\}$. As the domain Ω satisfies $H^{n-q}(\Omega, \mathcal{S}) = 0 = H^{n-q+1}(\Omega, \mathcal{S})$ for all coherent $\mathcal{O}_X$-modules $\mathcal{S}$, we can apply Theorem 1.1 twice for forms of different bidegree. More precisely for $N_0 + 2$ there exists $N_1 (>> N_0 + 2)$ such that if $F \in \mathcal{H}_{N_1}^{n-q,\,loc}(\Omega)$, $\overline{\partial}F = 0$ on $\Omega \setminus A$ then, there exists $a \in \mathcal{H}_{N_0+2}^{n-q-1,\,loc}(\Omega)$ satisfying $\overline{\partial}a = F$ on $\Omega \setminus A$. Similarly, for the above N_1 there exists an $N (>> N_1)$ such that if $G \in \mathcal{H}_N^{n-q+1,\,loc}(\Omega)$, $\overline{\partial}$-closed on $\Omega \setminus A$ then, there exists a $b \in \mathcal{H}_N^{n-q,\,loc}(\Omega)$ satisfying $\overline{\partial}b = G$ on $\Omega \setminus A$.

Let f, N_0 be as in Theorem 4.2 and N, N_1 be chosen as above. Consider the following map:

$$L_f : \mathcal{H}_N^{n-q+1,\,loc}(\Omega) \cap \operatorname{kern}(\overline{\partial}) \to \mathbb{C}$$

defined by

$$L_f(w) := (-1)^{p+q+1} \int_{\Omega\setminus A} v \wedge f$$

where $v \in \mathcal{H}_{N_1}^{n-q,\,loc}(\Omega)$ is a solution to $\overline{\partial}v = w$ on $\Omega \setminus A$ (such a solution always exist by Theorem 1.1).

First of all we need to show that L_f is well-defined, i.e. independent of the choice of the solution $v \in \mathcal{H}_{N_1}^{n-q,\,loc}$ to the equation $\overline{\partial}v = w$ on $\Omega \setminus A$. It suffices to show that $\int_{\Omega\setminus A} v \wedge f = 0$ when $v \in \mathcal{H}_{N_1}^{n-q,\,loc}(\Omega)$ and $\overline{\partial}v = 0$ on $\Omega \setminus A$. According to what was discussed above there exists an $a \in \mathcal{H}_{N_0+2}^{n-q-1,\,loc}(\Omega)$ satisfying $\overline{\partial}a = v$ on $\Omega \setminus A$.

Let $\chi_\nu \in C^\infty(\operatorname{Reg} X)$ be the smooth cut-off functions that were constructed in section 2.3. Recall that $0 \le \chi_\nu \le 1$, $\chi_\nu(z) = 1$ when $d_A(z) > \frac{1}{\nu}$, $\chi_\nu(z) = 0$ if $d_A(z) \le \frac{1}{2\nu}$ and such that for any compact $K \subset X$ we have $|\overline{\partial}\chi_\nu| \le C_K \nu$ on $\operatorname{Reg} X \cap K$, for some positive constant C_K independent of ν. Then $L_f(w) = \lim_{\nu\to\infty} \int_{\Omega\setminus A} \chi_\nu\, v \wedge f = \lim_{\nu\to\infty} \int_{\Omega\setminus A} \chi_\nu\, \overline{\partial} a \wedge f = -\lim_{\nu\to\infty} \int_{\Omega\setminus A} \overline{\partial}\chi_\nu \wedge a \wedge f$. The last equality follows from integration by parts arguments; the form $\chi_\nu\, f$ has compact support on $\Omega \setminus A$ but is not really smooth, so a smoothing argument is needed in order to apply the standard Stokes' theorem. Let $I_\nu := \int_{\Omega\setminus A} \overline{\partial}\chi_\nu \wedge a \wedge f$. By Cauchy-Schwarz inequality we have

$$|I_\nu|^2 \le \left(\int_{\operatorname{supp} f \cap \operatorname{supp}(\overline{\partial}\chi_\nu)} |d_A\, \overline{\partial}\chi_\nu|^2\, |f|^2\, d_A^{N_0}\, dV \right) \left(\int_{\{\chi_\nu < 1\} \cap \operatorname{supp} f} |a|^2\, d_A^{-(N_0+2)}\, dV \right) = A \cdot B$$

But B is finite and $A \to 0$ as $\nu \to \infty$ since $|d_A\, \overline{\partial}\chi_\delta| \le C$ and $\int_{\Omega\setminus A} |f|^2\, d_A^{N_0}\, dV < \infty$. Hence $\lim_{\nu\to\infty} I_\nu = 0$ and thus L_f is well-defined.

Clearly L_f is a linear map. By Cauchy-Schwarz we have

$$|L_f(w)| \le C_0 \left(\int_{\operatorname{supp} f} |v|^2\, d_A^{-N_1}\, dV \right)^{\frac{1}{2}} \left(\int_{\Omega\setminus A} |f|^2\, d_A^{N_0}\, dV \right)^{\frac{1}{2}} \tag{20}$$

To obtain this inequality we used the fact that $N_1 >> N_0$ and hence $d_A^{N_1 - N_0} \le C_0$ on $\overline{\Omega}$. We want to show that L_f factors into a bounded linear functional on a subspace $\mathcal{A}$ of a Hilbert space. Recall that the following lemma was proven in [4] using the open mapping theorem for Fréchet spaces.

Lemma 4.3. *(Lemma 4.2 in [4]) Let M be a complex manifold and let E and F be Fréchet spaces of differential forms (or currents) of type $(p, q-1)$, (p, q), whose topologies are finer (possibly strictly finer) than the weak topology of currents. Assume that for every $f \in F$, the equation $\overline{\partial} u = f$ has a solution $u \in E$. Then, for every continuous seminorm p on E, there is a continuous seminorm q on F such that the equation $\overline{\partial} u = f$ has a solution with $p(u) \le q(f)$ for every $f \in F$, $q(f) > 0$.*

Let $p(v) := \left(\int_{\operatorname{supp} f} |v|^2\, d_A^{-N_1}\, dV \right)^{\frac{1}{2}}$. Using the lemma in our situation, given the seminorm p there exists an open set $W \Subset \Omega$ (that depends on the $\operatorname{supp} f$) such that for all $w \in \mathcal{H}_N^{n-q+1,\, loc}(\Omega) \cap \operatorname{kern}(\overline{\partial})$ with $q(w) = \left(\int_W |w|^2\, d_A^{-N}\, dV \right)^{\frac{1}{2}} > 0$ there exists a solution v to $\overline{\partial} v = w$ on $\Omega \setminus A$ and a positive constant C satisfying

$$\left(\int_{\operatorname{supp} f} |v|^2\, d_A^{-N_1}\, dV \right)^{\frac{1}{2}} \le C\, q(w). \tag{21}$$

If $q(w) = 0$ then the same argument will imply that for every $\epsilon > 0$ there exists a solution v_ϵ to $\overline{\partial} v_\epsilon = w$ on $\Omega \setminus A$ with $p(v_\epsilon) < \epsilon$. Then for such a w we have: $|L_f(w)| \le C_0 \epsilon \int_{\Omega\setminus A} |f|^2\, d_A^{N_0}\, dV$. Here we used the fact that $L_f(w)$ is well-defined independent of the choice of solution $v \in \mathcal{H}_{N_1}^{n-q,\, loc}(\Omega)$. Taking the limit as $\epsilon \to 0$ we obtain

$$|L_f(w)| = 0 \le C_0\, q(w) \left(\int_{\Omega\setminus A} |f|^2\, d_A^{N_0}\, dV \right)^{\frac{1}{2}}. \tag{22}$$

Combining (20), (21), (22) we obtain for all $w \in \mathcal{H}_N^{n-q+1,\, loc} \cap \operatorname{kern}(\overline{\partial})$

$$|L_f(w)| \leq C_0\, C\, q(w) \left(\int_{\Omega \setminus A} |f|^2\, d_A^{N_0}\, dV \right)^{\frac{1}{2}}. \tag{23}$$

From (23) we see that $L_f(w)$ depends only on $w_{\upharpoonright W}$. Indeed, let $w,\, w' \in \mathcal{H}_N^{n-q+1,\, loc} \cap \operatorname{kern}(\overline{\partial})$ such that $w_{\upharpoonright W} = w'_{\upharpoonright W}$. Then $L_f(w) = L_f(w - w' + w') = L_f(w - w') + L_f(w')$. From (23) we obtain that $|L_f(w - w')| \leq C_0\, C\, q(w - w') \left(\int_{\Omega \setminus A} |f|^2\, d_A^{N_0}\, dV \right)^{\frac{1}{2}}$. But $q(w - w') = 0$ as $w - w' = 0$ on W. Hence $L_f(w - w') = 0$ and thus $L_f(w) = L_f(w')$. Hence L_f factors to a well-defined bounded linear functional on $\mathcal{A} := \{w_{\upharpoonright W};\quad w \in \mathcal{H}_N^{n-q+1,\, loc}(\Omega) \cap \operatorname{kern}(\overline{\partial})\} \subset \mathcal{H}_N^{n-q+1}(W)$. Here $\mathcal{H}_N^{n-q+1}(W) := \{w \in L^{2,\, loc}_{n-p,\, n-q+1}(W \setminus A) := \int_{W \setminus A} |F|^2\, d_A^{-N}\, dV < \infty\}$.

We make a norm-preserving extension of the above functional L_f to $\mathcal{H}_N^{n-q+1}(W)$. Let us call $\tilde{L}_f$ the extended functional. By Riesz representation theorem there exists a $u' \in \mathcal{H}_N^{n-q+1}(W)$ such that for all $w \in \mathcal{H}_N^{n-q+1}(W)$ we have

$$\tilde{L}_f(w) = \int_{W \setminus A} < w,\, u' >\; d_A^{-N}\, dV. \tag{24}$$

Set $u := d_A^{-N}\, \overline{*}\, u'$ on $W \setminus A$ (here $*$ is the Hodge-star operator) and extend by zero outside $\overline{W}$. We claim that u is the desired solution of Theorem 5.3. Certainly $\operatorname{supp} u \Subset \Omega$ and $\int_{\Omega \setminus A} |u|^2\, d_A^N\, dV = \int_{W \setminus A} |u'|^2\, d_A^{-N}\, dV < \infty$, since $u' \in \mathcal{H}_N^{n-q+1}(W)$. We can control the weighted L^2 norm of u' in terms of the weighted L^2-norm of f taking into consideration the following:

$$\left(\int_{W \setminus A} |u'|^2\, d_A^{-N}\, dV \right)^{\frac{1}{2}} = \|\tilde{L}_f\| = \|L_{f\,\upharpoonright \mathcal{A}}\| = \tag{25}$$

$$= \sup\{|L_f(w)| :\ w \in \mathcal{A} \text{ with } q(w) \leq 1\} \leq C_0\, C \left(\int_{\Omega \setminus A} |f|^2\, d_A^{N_0}\, dV \right)^{\frac{1}{2}}.$$

It remains to show that $\overline{\partial} u = f$ on $\Omega \setminus A$. Let $\phi \in C^\infty_{0,\, (n-p,\, n-q)}(\Omega \setminus A)$ be a smooth compactly supported form of bidegree $(n-p,\, n-q)$ on $\Omega \setminus A$. We need to show that

$$\int_{\Omega \setminus A} \overline{\partial}\phi \wedge u = (-1)^{p+q+1} \int_{\Omega \setminus A} \phi \wedge f. \tag{26}$$

But $\phi \in \mathcal{H}_{N_1}^{n-q,\, loc}(\Omega)$, $\overline{\partial}\phi \in \mathcal{H}_N^{n-q+1,\, loc}(\Omega)$ and $\overline{\partial}\phi_{\upharpoonright W} \in \mathcal{A}$. Therefore from the definition of L_f we have that

$$\tilde{L}_f(\overline{\partial}\phi_{\upharpoonright W}) = L_f(\overline{\partial}\phi_{\upharpoonright W}) = (-1)^{p+q+1} \int_{\Omega \setminus A} \phi \wedge f$$

On the other hand from (24) we have that

$$\tilde{L}_f(\overline{\partial}\phi_{\upharpoonright W}) = \int_{W \setminus A} \overline{\partial}\phi \wedge u = \int_{\Omega \setminus A} \overline{\partial}\phi \wedge u.$$

Putting the last two equalities together we obtain (26).

By the definition of u and (25) we can obtain the following estimate for the solution u:

$$\int_{\Omega\setminus A} |u|^2\, d_A^N\, dV \le \tilde{C} \int_{\Omega\setminus A} |f|^2\, d_A^{N_0}\, dV$$

where $\tilde{C}$ is a positive constant that depends on N, N_0, Ω, $\operatorname{supp} f$.

Remark: We can be more precise about the dependence of $\operatorname{supp} u$ and the constant $\tilde{C}$ (that appears in the last inequality) on $\operatorname{supp} f$. Let X, Ω be as in theorem 4.2 and let N_0 be a non-negative integer. There exists a positive integer N that depends on N_0 and Ω such that the following is true: For every compact $K \subset \Omega$, there exists a compact $K' \subset \Omega$ and a positive constant C that depends on K, N, N_0, Ω such that for every (p,q) form f with $\operatorname{supp} f \subset K$ and $\int_{K\setminus A} |f|^2\, d_A^{N_0}\, dV < \infty$, $\overline{\partial}$-closed on $\Omega \setminus A$, there exists a solution u to $\overline{\partial} u = f$ on $\Omega\setminus A$ with $\operatorname{supp} u \subset K'$ satisfying

$$\int_{K'\setminus A} |u|^2\, d_A^N\, dV \le \tilde{C} \int_{K\setminus A} |f|^2\, d_A^{N_0}\, dV.$$

The proof follows along the same lines as the proof of Theorem 4.2, by taking as $p(v) := \left(\int_{K\setminus A} |v|^2\, d_A^{-N_1}\, dV\right)^{\frac{1}{2}}$. Then there exist an open, relatively compact subset W of Ω (that depends on K) a seminorm $q(w) := \int_{W\setminus A} |w|^2\, d_A^{-N}\, dV$ and a positive constant C' (that depends on K) such that the equation $\overline{\partial} v = w$ has a solution v satisfying $p(v) \le C'\, q(w)$. The rest of the proof of Theorem 4.2 carries over. The compact K' is chosen to be $K' := \operatorname{closure}(W)$ and the constant $\tilde{C} = C'\, C_0$ where C_0 depends on N, N_0, Ω.

5. An analytic proof of Theorem 1.3

In [14] the following proposition was proved:

Proposition 5.1. *(Proposition 5.1 in* [14]*) Let X be a connected, non-compact normal complex space and K a compact subset of X. Let us assume that K has an open neighborhood Ω in X with the following property (P)*

For every "nice" $\overline{\partial}$-closed $(0,1)$-form f on $Reg\,\Omega$ with $supp_X\, f := closure_X\,(supp\, f)$ compact in Ω, the equation $\overline{\partial} u = f$ has a solution on $Reg\,\Omega$ with $supp_X\, u := closure_X\,(supp\, u)$ compact in Ω.

Then, for every open neighborhood D of K with $D\setminus K$ connected and every $s \in \Gamma(D\setminus K, \mathcal{O})$ there exists a unique $\tilde{s} \in \Gamma(D, \mathcal{O})$ such that $\tilde{s} = s$ on $D \setminus K$.

"Nice" in the above statement means that the form f can be smoothly extended on U for some local embedding $\phi : W \subset \Omega \to U^{\text{open}} \subset \mathbb{C}^N$.

Hence to prove Theorem 1.3 it remains to show that we can always find an open neighborhood Ω of K that satisfies property (P) of the above proposition. Since K is compact on an $(n-1)$-complete space we can always find an open, relatively compact $(n-1)$-complete subdomain Ω of X that contains K. Property (P) for such domains has been established by Theorem 4.2 (for $A := \operatorname{Sing} X$) and the proof of Theorem 1.3 is completed.

6. A generalization of a lemma in [4]

Proposition 2.2 in section 2 was a special case of the following key lemma in [4]:

Lemma 6.1. *(Lemma 2.1 in* [4]*) For each $q > 0$ and for each coherent, torsion free $\mathcal{O}_{\tilde{X}}$-module $\mathcal{S}$ there exists a $T \in \mathbb{N}$ such that $i_{\tilde{\Omega},*} : H^q(\tilde{\Omega}, J^T\mathcal{S}) \to H^q(\tilde{\Omega}, \mathcal{S})$ is the zero map, where $i : J^T\mathcal{S} \hookrightarrow \mathcal{S}$ is the inclusion map.*

In this section we shall see that it is possible to drop the assumption that S is torsion free and J is a principal ideal sheaf in the lemma above. Hence lemma 6.1 should be valid for any proper modification $\pi : \widetilde{X} \to X$ where is X a pure n-dimensional reduced Stein space, Ω open relatively compact Stein subdomain of $X, \widetilde{\Omega} =: \pi^{-1}(\Omega)$, S is any coherent $\mathcal{O}_{\widetilde{X}}$-module and where J is the ideal sheaf of the exceptional set $\tilde{A}$ of the proper modification π.

Remark: Recall that a proper modification between two complex spaces $\widetilde{X}$, X consists of a proper surjective holomorphic map $\pi : \widetilde{X} \to X$, closed analytic subsets $\widetilde{A}$, A of $\widetilde{X}$, X respectively, such that a) $A = \pi(\widetilde{A})$, b) $\widetilde{A}$, A are analytically rare, c) $\pi_{\restriction \widetilde{X}\setminus\widetilde{A}} : \widetilde{X} \setminus \widetilde{A} \to X \setminus A$ is a biholomorphism and d) $\widetilde{A}$, A are minimal with respect to properties a)-c). For reduced spaces X saying that a closed analytic set A is analytically rare is equivalent to saying that no connected component of X is contained in A. We would be primarily interested in proper modifications $\pi : \widetilde{X} \to X$, where X is a reduced, pure n-dimensional Stein space. Then $\widetilde{X}$ would also be reduced and pure n-dimensional (see Chapter VII, page 287-288 in [8]). If Ω is an open relatively compact Stein subdomain of X, then we know that Ω does not contain any compact n-dimensional irreducible component. Hence $\widetilde{\Omega}$ can not contain any compact n-dimensional irreducible component as well.

The new ingredient in the proof is based on an application of the Artin-Rees lemma:

Lemma 6.2. *(Artin-Rees Lemma, page 441 in* [12]*) Let M be a finitely generated module over a noetherian ring R, and let U be a submodule of M and I an ideal of R. Then there exists a positive integer k such that for all integers $n \geq k$ we have*

$$I^n\, M \cap U = I^{n-k}\,(I^k\, M \cap U).$$

For completion we shall recall the proof of lemma 6.1 and mention all the necessary modifications.

Proof of the more general version of Lemma 6.1. We shall prove lemma 6.1 (under the more general assumptions on π, $\mathcal{S}$, J mentioned above) using downward induction on $q > 0$. Observe that $\tilde{\Omega}$ is a pure n-dimensional complex space with no compact n-dimensional branches. It follows from the Main Theorem in Siu [16] that $H^n(\tilde{\Omega}, \mathcal{S}) = 0$ for every coherent $\mathcal{O}_{\tilde{X}}$-module $\mathcal{S}$. Hence, the statement is true for $q = n$ and any $T \in \mathbb{N}$.

When $q > 0$, $\mathrm{Supp} R^q\pi_*\mathcal{S}$ is contained in A. The annihilator ideal $\mathcal{A}'$ of $R^q\pi_*\mathcal{S}$ is coherent and by Cartan's Theorem A there exist functions $f_1, \cdots, f_L \in \mathcal{A}'(X)$ that generate each stalk $\mathcal{A}'_z$ in a neighborhood of $\overline{\Omega}$. Let $\mathcal{A}$ be the $\mathcal{O}_{\tilde{X}}$-ideal generated by $\tilde{f}_j = f_j \circ \pi$, $1 \leq j \leq L$. A crucial observation which will be useful later, is that $(\tilde{f}_j)_{\tilde{\Omega},*} : H^q(\tilde{\Omega}, \mathcal{S}_{|_{\tilde{\Omega}}}) \to H^q(\tilde{\Omega}, \mathcal{S}_{|_{\tilde{\Omega}}})$ are zero for all j, $1 \leq j \leq L$, $q > 0$. To see this, consider the following commutative diagram

$$\begin{array}{ccc} H^q(\tilde{\Omega}, \mathcal{S}_{|_{\tilde{\Omega}}}) & \xrightarrow{(\tilde{f}_j)_{\tilde{\Omega},*}} & H^q(\tilde{\Omega}, \mathcal{S}_{|_{\tilde{\Omega}}}) \\ \cong\big\downarrow & & \cong\big\downarrow \\ R^q\pi_*\mathcal{S}(\Omega) & \xrightarrow{(f_j)_{\Omega,\#}} & R^q\pi_*\mathcal{S}(\Omega) \end{array}$$

The vertical maps are isomorphisms, due to Satz 5, Section 2, in [5]. Recalling the way $\mathcal{O}_X$ acts on $R^q\pi_*\mathcal{S}$ and using the fact that the f_j's are in the annihilator ideal of $R^q\pi_*\mathcal{S}$ we conclude that $(f_j)_{\Omega,\#} = 0$. Hence, due to the commutativity of the above diagram $(\tilde{f}_j)_{\tilde{\Omega},*}$ is zero.

Let $Z(\mathcal{A})$ (resp. $Z(\mathcal{A}')$) denote the zero variety of $\mathcal{A}$ (resp. $\mathcal{A}'$). Since $Z(\mathcal{A}') = \mathrm{Supp} R^q\pi_*\mathcal{S}$ is contained in A, we have that $Z(\mathcal{A})$ is contained in $\tilde{A}$ near $\overline{\tilde{\Omega}}$. Thus by Rückert's Nullstellensatz for ideal sheaves, (see Theorem, page 82 in [9]), we have $J^\mu \subset \mathcal{A}$ on $\tilde{\Omega}$ for some $\mu \in \mathbb{N}$. Consider the surjection $\phi : \mathcal{S}^{\oplus L} \to \mathcal{A}\,\mathcal{S}$ given by $(s_1, \cdots, s_L) \to \sum_1^L \tilde{f}_j s_j$ and set $K = \ker\phi$. By definition the sequence

$$0 \to K \xrightarrow{i} \mathcal{S}^{\oplus L} \xrightarrow{\phi} \mathcal{A}\,\mathcal{S} \to 0 \tag{27}$$

is exact. Now, unlike the proof of Lemma 2.1 in [4] the following sequence will no longer be exact

$$0 \to J^a\, K \xrightarrow{i} J^a\, \mathcal{S}^{\oplus L} \xrightarrow{\phi} J^a \mathcal{A}\,\mathcal{S} \to 0.$$

Recall that in our situation S is not torsion free and J is not locally generated by one element. On the other hand for every $a \geq 0$ the morphism $\phi : J^a\, \mathcal{S}^{\oplus\, L} \to J^a\, \mathcal{A}\,\mathcal{S}$ is still surjective. Let $K_a := \mathrm{kern}(\phi)_{|_{J^a\, \mathcal{S}^{\oplus\, L}}} = K \cap J^a\, \mathcal{S}^{\oplus\, L}$. Then the following sequence will be exact

$$0 \to K_a \xrightarrow{i} J^a\, \mathcal{S}^{\oplus L} \xrightarrow{\phi} J^a \mathcal{A}\,\mathcal{S} \to 0.$$

Taking all the above into consideration we obtain the following commutative diagram:

$$\begin{tikzcd}
 & H^q(\tilde{\Omega}, J^{a+\mu}\mathcal{S}) \arrow[d] & \\
 & H^q(\tilde{\Omega}, J^a\mathcal{A}\,\mathcal{S}) \arrow[r, "\delta"] \arrow[d] & H^{q+1}(\tilde{\Omega}, K_a) \arrow[d, "i_1"'] \\
H^q(\tilde{\Omega}, \mathcal{S})^{\oplus L} \arrow[r, "\phi_{\tilde{\Omega},*}"] \arrow[dr, "\chi"] & H^q(\tilde{\Omega}, \mathcal{A}\,\mathcal{S}) \arrow[r, "\delta"] \arrow[d, "i_2"] & H^{q+1}(\tilde{\Omega}, K) \\
 & H^q(\tilde{\Omega}, \mathcal{S}) &
\end{tikzcd}$$

where the third row is exact (as part of the long exact cohomology sequence that arises from (27)) and the vertical maps are induced by sheaf inclusions. The map χ is defined to be $\chi := i_2 \circ \phi_{\tilde{\Omega},\,*}$ and we can show that $\chi(c_1, \cdots, c_L) = \sum_{j=1}^{L} (\tilde{f}_j)_{\tilde{\Omega},\,*} c_j$, where $c_j \in H^q(\tilde{\Omega}, \mathcal{S}),\ 1 \leq j \leq L$.

In the original commutative diagram in [4] the cohomology group in the second line-third column was $H^{q+1}(\tilde{\Omega},\, J^a\, K)$. The induction hypothesis applied to K, allowed us to claim that there exists an a large enough so that $i_1 = 0$. Here, we *need* to show that $i_1 : H^{q+1}(\tilde{\Omega},\, K_a) \to H^{q+1}(\tilde{\Omega},\, K)$ is the zero map assuming by the induction hypothesis that the map $H^{q+1}(\tilde{\Omega},\, J^\alpha\, K) \xrightarrow{I} H^{q+1}(\tilde{\Omega},\, K)$ is the zero map for some α sufficiently large. If we were able to show that the map i_1 factors through $H^{q+1}(\tilde{\Omega},\, J^a\, K)$ then we would be done since the rest of the proof carries over verbatim.

For each positive integer k we set

$$A_k := \mathrm{supp}\, \left(\frac{(J^{\alpha+k}\mathcal{S}^{\oplus L} \cap K)}{J^\alpha\,(J^k\, \mathcal{S}^{\oplus L} \cap K)} \right).$$

Each A_k is a subvariety of $\widetilde{X}$ and by Artin-Rees lemma we have that $\cap_{k=1}^{\infty} A_k = \emptyset$. By the descending chain property of analytic varieties, there exists an integer m such that $\tilde{\Omega} \cap (\cap_{k=1}^{m} A_k) = \emptyset$, since $\tilde{\Omega} \Subset \tilde{X}$. Hence for every $x \in \tilde{\Omega}$ there exists $k_x \leq m$ such that

$$K_{k_x+\alpha,\,x} = J_x^{k_x+\alpha}\, \mathcal{S}_x^{\oplus L} \cap K_x = J_x^\alpha\, (J_x^{k_x}\, \mathcal{S}_x^{\oplus L} \cap K_x) \subset J_x^\alpha\, K_x.$$

Let us choose $a := m + \alpha$. Then we have $K_{a,\,x} \subset K_{k_x+\alpha,\,x} \subset J_x^a\, K_x$. Thus over $\tilde{\Omega}$, the inclusion $K_a \to K$ factors via $J^\alpha\, K$ so the map $i_1 : H^{q+1}(\tilde{\Omega},\, K_a) \to H^{q+1}(\tilde{\Omega},\, K)$ is the zero map.

Then, for an element $\sigma \in H^q(\tilde{\Omega}, \mathcal{A}\,\mathcal{S})$ that comes from $H^q(\tilde{\Omega}, J^{a+\mu}\mathcal{S})$, we have $\delta\sigma = 0$, so $\sigma = \phi_{\tilde{\Omega},*}(\sigma_1, \cdots, \sigma_L)$ for $\sigma_j \in H^q(\tilde{\Omega}, \mathcal{S})$, $1 \leq j \leq L$. By the crucial observation above and the way χ is defined, we conclude that

χ is the zero map. Hence $i_2(\sigma) = i_2 \circ \phi_{\tilde{\Omega},*}(\sigma_1, \cdots, \sigma_L) = \sum_{j=1}^{L} (\tilde{f}_j)_{\tilde{\Omega},*}\sigma_j = 0$. Thus, for $i : J^{a+\mu}\mathcal{S} \hookrightarrow \mathcal{S}$ the inclusion map, we have that $i_{\tilde{\Omega},*} : H^q(\tilde{\Omega}, J^{a+\mu}\mathcal{S}) \to H^q(\tilde{\Omega}, \mathcal{S})$ is the zero map.

7. Appendix

The purpose of this appendix is to prove the equivalence of the corresponding distance functions d_A over K that was alluded to in section 2.1., if we choose different compacts $K' \Supset K$ and different embeddings.

Lemma 7.1. *Let $\phi : U \to \mathbb{C}^T$ be a local embedding and let K, L, M be compact subsets of U such that $K \Subset L \Subset U$ and $K \Subset M \Subset U$. Let $d_1(x) := dist(\phi(x), \phi(L \cap A))$ and $d_2(x) := dist(\phi(x), \phi(M \cap A))$. Then*

$$d_1(x) \sim_K d_2(x).$$

Proof: If $M \subset L$ then clearly $d_1(x) \leq d_2(x)$. Suppose that $M \not\subset L$. Then $\text{dist}(\phi(K), \phi(M \setminus \overset{0}{L} \cap A)) = a > 0$ (here $\overset{0}{L}$ denotes the interior of the set L). Let us assume that there existed $x \in K$ such that $d_2(x) < d_1(x)$. Then we must have that $d_2(x) \geq a$, hence $1 \leq a^{-1} d_2(x)$. Let $B := \max_{x \in K} d_1(x)$. Then we have

$$d_1(x) \leq B = B \times 1 \leq B\, a^{-1} d_2(x).$$

Choosing as $C_1 := \max(\{1, B\, a^{-1}\})$ we obtain for all $x \in K$ that $d_1(x) \leq C_1\, d_2(x)$. Similarly we can prove that there exists a constant $C_2 \geq 1$ such that $d_2(x) \leq C_2\, d_1(x)$ for all $x \in K$. Choosing as $C := \max(C_1, C_2)$ we obtain the equivalence of d_1 and d_2 over K.

Lemma 7.2. *Let $\phi_i : U_i \to \mathbb{C}^{T_i}$ be local embeddings for $i = 1, 2$ and let K, K' be a compact subsets of $U_1 \cap U_2$ such that $K \Subset K' \Subset U_1 \cap U_2$. Then we have:*

$$dist(\phi_1(x), \phi_1(K' \cap A)) \sim_K dist(\phi_2(x), \phi_2(K' \cap A)).$$

Proof: Let $x_0 \in K$ and (ϕ_0, V_0, W_0) be a minimal embedding of X at x_0; $\phi_0 : V_0 \to W_0 \subset \mathbb{C}^{N_0}$. In this case we know that $d\,\phi_{0,\,x_0} : T_{x_0}X \to T_{\phi_0(x_0)}W_0$ is an isomorphism. By shrinking V_0 we can assume that $V_0 \subset U_1 \cap U_2$. Shrinking further V_0, W_0 and after a holomorphic change of coordinates we can assume that $\phi_0(x_0) = 0 = \phi_i(x_0)$ for $i = 1, 2$.

Claim: The maps $\phi_i \circ \phi_0^{-1} : \phi_0(V_0) \to \phi_i(V_0)$ for $i = 1, 2$ extend from $\phi_0(V_0)$ to an embedding $\tilde{\phi_i} : \mathbb{C}^{N_0} \to \mathbb{C}^{T_i}$.

Proof of Claim: Indeed, as ϕ_i is an embedding of X at x_0 we know that $d\,\phi_{i,\,x_0} : T_{x_0}X \to T_0\mathbb{C}^{T_i}$ is injective. Let $\phi_i \circ \phi_0^{-1} : \phi_0(V_0) \to \phi_i(V_0)$ be given by the coordinate map $(f_1, \cdots, f_{T_i})$. Then $\phi_i \circ \phi_0^{-1}$ extends to a map $\tilde{\phi_i} : \mathbb{C}^{N_0} \to \mathbb{C}^{T_i}$ by extending each f_j for $j = 1, \cdots, T_i$. Then we have the following commutative diagram

$$\begin{tikzcd}
T_{x_0}X \arrow[r, "d\phi_{i,\,x_0}"] \arrow[d, "d\,\phi_{0,\,x_0}"', "\cong"] & T_0\mathbb{C}^{T_i} \\
T_0\mathbb{C}^{N_0} \arrow[ur, "d\tilde{\phi}_{i,\,0}"'] &
\end{tikzcd}$$

Taking into account the construction of the $\tilde{\phi_i}$ the fact that $d\phi_0$ is an isomorphism at x_0 and each $d\phi_i$ is an embedding of X at x_0 we obtain that $d\tilde{\phi_i}_{x_0}$ is injective; hence $\tilde{\phi_i}$ is an embedding at x_0 for $i = 1, 2$.

When $L_0 \Subset W_0$ is a neighborhood of $\phi_0(x_0) = 0$ it follows that

$$|\tilde{\phi_i}(x) - \tilde{\phi_i}(y)| \sim_K |x - y|$$

on $L_0 \times L_0$ and for $i = 1, 2$. Hence, when K_0 is a compact neighborhood of x_0 contained in $\overset{0}{K'}_0 := {\phi_0}^{-1}(\overset{0}{L}_0)$ ($\overset{0}{L}_0$ or $\overset{0}{K'}_0$ denotes the interior of the corresponding sets) then we have

$$\mathrm{dist}(\phi_i(x),\, \phi_i(K_0' \cap A)) \sim_{K_0} \mathrm{dist}(\phi_0(x),\, \phi_0(K_0' \cap A)).$$

Using the transitivity of $\sim_{K_0}$ and Lemma 7.1 we obtain that

$$\mathrm{dist}(\phi_1(x),\, \phi_1(K' \cap A)) \sim_{K_0} \mathrm{dist}(\phi_2(x),\, \phi_2(K' \cap A)).$$

We can now cover K by finitely many K_0 and obtain the desired result.

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